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ON THE STATE OF STRESS AND STRAIN OF STOCHASTICALLY NONHOMOGENEOUS ELASTIC BODIES

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We consider the problem of the determination of the state of stress and strain of stochastically nonhomogenous elastic bodies. As a result of the application of the generalized method of statistical linearization, we obtain a closed system of equations and for a normal probability distribution a system of integro-differential equations relative to the mathematical expectations.

For an isotropic medium, whose elastic moduli are homogeneous isotropic random functions of the coordinates, we obtain the expressions of the effective elasticity moduli in terms of the mathematical expectations and the dispersion of the random functions describing the properties of the medium.

1. We assume that the random functions are connected by the nonlinear relation

$$Z = X_{ij} Y_{ij} \tag{1.1}$$

Here and in the sequel, repeated Latin indices indicate summation from one to three. We represent the functions which occur in (1.1) in the form of a sum of mathematical expectations and deviations from them

$$Z = \langle Z \rangle + Z', \qquad X_{ij} = \langle X_{ij} \rangle + X_{ij}', \ Y_{ij} = \langle Y_{ij} \rangle + Y'_{ij}$$

The approximating function of the relation (1.1) is taken in the form

$$V = a + a_{ij} Y_{ij}' + b_{ij} X_{ij}' \tag{1.2}$$

The nonrandom functions a, a_{ij}, b_{ij} will be defined from the condition of the minimum mathematical expectation of the square of the difference between the correct and the approximating functions [1]

$$\langle (Z - V)^2 \rangle = \min \tag{1.3}$$

Applying the extremum condition to the expression (1.3) with respect to the parameters, we obtain the linear system of equations with respect to the unknowns

$$\begin{aligned} & a = \langle X_{ij} \rangle \langle Y_{ij} \rangle + \langle X_{mn}' Y_{mn}' \rangle \\ & a_{ij} \langle Y_{mn}' Y_{ij}' \rangle + b_{ij} \langle Y_{mn}' X_{ij}' \rangle = \langle X_{ij} \rangle \langle Y_{ij}' Y_{mn}' \rangle + \\ & \quad \langle Y_{ij} \rangle \langle X_{ij}' Y_{mn}' \rangle + \langle X_{ij}' Y_{ij}' Y_{mn}' \rangle \\ & a_{ij} \langle Y_{ij}' X_{mn}' \rangle + b_{ij} \langle X_{ij}' X_{mn}' \rangle = \langle X_{ij} \rangle \langle Y_{ij}' X_{mn}' \rangle + \\ & \quad \langle Y_{ij} \rangle \langle X_{ij}' X_{mn}' \rangle + \langle X_{ij}' Y_{ij}' X_{mn}' \rangle \end{aligned} \tag{1.4}$$

2. We consider a nonhomogeneous anisotropic elastic body in which the stresses σ_{ij} and the strains ε_{ij} are connected by the generalized Hooke's law with the tensor of elastic moduli c_{ijmn}, defining the random tensor field

$$\sigma_{ij} = c_{ijmn}\varepsilon_{mn}, \quad \sigma_{ij,j} = 0, \qquad \varepsilon_{mn} = {}^1/_2\,(u_{m,n} + u_{n,m}) \tag{2.1}$$

Carrying out the statistical linearization (1.2) –(1.4) of the system (2.1), we obtain the following system of equations:

$$\langle\sigma_{ij}\rangle = \langle c_{ijmn}\rangle\langle\varepsilon_{mn}\rangle + \langle c'_{ijmn}\,\varepsilon'_{mn}\rangle \tag{2.2}$$
$$\langle\sigma_{ij,j}\rangle = 0, \qquad \langle\varepsilon_{mn}\rangle = {}^1/_2\,(\langle u_{m,n}\rangle + \langle u_{n,m}\rangle)$$

$$\sigma'_{ij} = a_{ijmn}\varepsilon'_{mn} + b_{mn}\,c'_{ijmn} \tag{2.3}$$
$$\sigma'_{ij,j} = 0, \qquad \varepsilon'_{mn} = {}^1/_2\,(u'_{m,n} + u'_{n,m})$$

$$a_{\alpha\beta ij}\ \langle\varepsilon'_{ij}c'_{\alpha\beta mn}\rangle + b_{ij}\ \langle c'_{\alpha\beta ij}\,c'_{\alpha\beta mn}\rangle = \tag{2.4}$$
$$\langle c_{\alpha\beta ij}\rangle\ \langle\varepsilon'_{ij}c'_{\alpha\beta mn}\rangle + \langle\varepsilon_{ij}\rangle\ \langle c'_{\alpha\beta ij}\ c'_{\alpha\beta mn}\rangle + \langle c'_{\alpha\beta ij}c'_{\alpha\beta mn}\varepsilon'_{ij}\rangle$$
$$a_{\alpha\beta ij}\langle\varepsilon'_{ij}\varepsilon'_{mn}\rangle + b_{ij}\ \langle c'_{\alpha\beta ij}\ \varepsilon'_{mn}\rangle =$$
$$\langle c_{\alpha\beta ij}\rangle\ \langle\varepsilon'_{ij}\ \varepsilon'_{mn}\rangle \ + \langle\varepsilon_{ij}\rangle\ \langle c'_{\alpha\beta ij}\ \varepsilon'_{mn}\rangle + \langle c'_{\alpha\beta ij}\varepsilon'_{ij}\varepsilon'_{mn}\rangle$$

The repeated Greek indices do not indicate summation. The system of equations (2.2)–(2.4) is closed. It can be solved more easily by the method of successive approximations [2].

The problem of the determination of the states of stress and strain is considerably simplified if the probability distribution is normal, which is often the case in linearelastic systems [3]. In this case from the equations (2.4) we obtain

$$a_{ijmn} = \langle c_{ijmn}\rangle, \quad b_{mn} = \langle\varepsilon_{mn}\rangle \tag{2.5}$$

Taking into account (2.5), equation (2.3) becomes

$$\langle c_{ijmn}\rangle u'_{m,nj} = -f_{ij,j}, \ f_{ij} = c'_{ijmn}\,\langle\varepsilon_{mn}\rangle \tag{2.6}$$

For a body of volume v with specified determined boundary conditions on its surface, the solution of the system (2.6) expressed by means of the Green's function, has the form [4]

$$u'_i = \int_v G_{in}(x, x')\,\frac{\partial f_{nj}(x')}{\partial x'_j}\,dv' \tag{2.7}$$

The first two relations of the system (2.2), after substituting into them the expression (2.7) and carrying out the operation of mathematical expectation, reduce to the following:

$$\langle c_{ijmn}\rangle\,\langle\varepsilon_{mn,j}\rangle + \frac{\partial}{\partial x_j}\int_v F_{qmn}\,\frac{\partial}{\partial x'_k}\ (\langle\varepsilon_{st}(x')\rangle\,c^{ijqm}_{nkst})\,dv' = 0 \tag{2.8}$$

$$c^{ijqm}_{nkst} = \langle c'_{nkst}(x)\,c'_{ijqm}(x')\rangle, \qquad F_{qmn} = \frac{\partial G_{qm}}{\partial x_n} + \frac{\partial G_{qn}}{\partial x_m}$$

Adjoining to (2.8) the last relation from (2.2), we obtain a closed system of integro-differential equations relative to the mean values of the displacement components.

3. We consider a stochastically nonhomogenous isotropic elastic body whose dimensions are much larger than the dimensions of the nonhomogeneities. We assume that the elastic moduli are homogeneous isotropic functions of the coordinates and that the body is in a macroscopic homogeneous deformed state, i.e.

$$c_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{3.1}$$
$$\langle\lambda\rangle = \text{const}, \quad \langle\mu\rangle = \text{const}, \quad \langle\varepsilon_{mn}\rangle = \text{const}$$

In this case, the equations corresponding to the systems (2.2) – (2.4) obtain the form

$$\langle\sigma_{ij}\rangle = 2\left(\langle\mu\rangle\langle\varepsilon_{ij}\rangle + \langle\mu'\varepsilon_{ij}'\rangle\right) + \delta_{ij}\left(\langle\lambda\rangle\langle\varepsilon_{nn}\rangle + \langle\lambda'\varepsilon_{nn}'\rangle\right) \tag{3.2}$$
$$\langle\sigma_{ij,j}\rangle = 0, \quad \langle\varepsilon_{ij}\rangle = {}^1/_2\left(\langle u_{i,j}\rangle + \langle u_{j,i}\rangle\right)$$
$$\sigma_{ij}' = 2\left(a_1\varepsilon_{ij}' + b_{ij}\mu'\right) + \delta_{ij}\left(a_1\varepsilon_{nn}' + b_2\lambda'\right) \tag{3.3}$$
$$\sigma_{ij,j}' = 0, \quad \varepsilon_{ij}' = {}^1/_2\left(u_{i,j}' + u_{j,i}'\right)$$
$$\begin{aligned}
&2\left(a_1\langle\varepsilon_{\alpha\beta}'\varepsilon_{\alpha\beta}'\rangle + b_{\alpha\beta}\langle\varepsilon_{\alpha\beta}'\mu'\rangle\right) + \delta_{\alpha\beta}\left(a_2\langle\varepsilon_{nn}'\varepsilon_{\alpha\beta}'\rangle + b_2\langle\lambda'\varepsilon_{\alpha\beta}'\rangle\right) = \\
&\quad 2\,\langle\mu\rangle\langle\varepsilon_{\alpha\beta}'\varepsilon_{\alpha\beta}'\rangle + \langle\varepsilon_{\alpha\beta}\rangle\langle\varepsilon_{\alpha\beta}'\mu'\rangle + \langle\varepsilon_{\alpha\beta}'\varepsilon_{\alpha\beta}'\mu'\rangle) + \\
&\quad \delta_{\alpha\beta}\left(\langle\lambda\rangle\langle\varepsilon_{\alpha\beta}'\varepsilon_{nn}'\rangle + \langle\varepsilon_{nn}'\rangle\langle\lambda'\varepsilon_{\alpha\beta}'\rangle + \langle\varepsilon_{nn}'\lambda'\varepsilon_{\alpha\beta}'\rangle\right) \\
&2\left(a_1\langle\mu'\varepsilon_{\alpha\beta}'\rangle + b_{\alpha\beta}\langle\mu'\mu'\rangle\right) + \delta_{\alpha\beta}\left(a_2\langle\varepsilon_{nn}'\mu'\rangle + b_2\langle\mu'\lambda'\rangle\right) = \\
&\quad 2\left(\langle\mu\rangle\langle\mu'\varepsilon_{\alpha\beta}'\rangle + \langle\varepsilon_{\alpha\beta}\rangle\langle\mu'\mu'\rangle + \langle\varepsilon_{\alpha\beta}\mu'\lambda'\rangle\right) + \\
&\quad \delta_{\alpha\beta}\left(\langle\lambda\rangle\langle\varepsilon_{nn}'\mu'\rangle + \langle\varepsilon_{nn}\rangle\langle\lambda'\mu'\rangle + \langle\varepsilon_{nn}\mu'\lambda'\rangle\right) \\
&2\left(a_1\langle\varepsilon_{nn}'\varepsilon_{\alpha\alpha}'\rangle + b_{\alpha\alpha}\langle\mu'\varepsilon_{nn}'\rangle\right) + a_2\langle\varepsilon_{nn}'\varepsilon_{mm}'\rangle + b_2\langle\varepsilon_{nn}'\lambda'\rangle = \\
&\quad 2\,\langle\mu\rangle\langle\varepsilon_{nn}'\varepsilon_{\alpha\alpha}'\rangle + \langle\varepsilon_{\alpha\alpha}\rangle\langle\mu'\varepsilon_{nn}'\rangle + \langle\varepsilon_{\alpha\alpha}'\mu'\varepsilon_{nn}'\rangle) + \\
&\quad \langle\lambda\rangle\langle\varepsilon_{nn}'\varepsilon_{ii}'\rangle + \langle\varepsilon_{nn}\rangle\langle\lambda'\varepsilon_{kk}'\rangle + \langle\varepsilon_{nn}'\varepsilon_{ii}'\lambda'\rangle \\
&2\left(a_1\langle\lambda'\varepsilon'_{\alpha\alpha}\rangle + b_{\alpha\alpha}\langle\mu'\lambda'\rangle\right) + a_2\langle\varepsilon_{nn}'\lambda'\rangle + b_2\langle\lambda'\lambda'\rangle = \\
&\quad 2\left(\langle\mu\rangle\langle\lambda'\varepsilon'_{\alpha\alpha}\rangle + \langle\varepsilon_{\alpha\alpha}\rangle\langle\lambda'\mu'\rangle + \langle\varepsilon_{\alpha\alpha}\mu'\lambda'\rangle\right) + \\
&\quad \langle\lambda\rangle\langle\varepsilon_{nn}'\lambda'\rangle + \langle\varepsilon_{nn}\rangle\langle\lambda'\lambda'\rangle + \langle\varepsilon_{nn}'\lambda'\lambda'\rangle
\end{aligned} \tag{3.4}$$

In the assumption that the probability distribution is normal, the solution of the system of equations (3.4) is

$$a_1 = \langle\mu\rangle, \quad b_{ij} = \langle\varepsilon_{ij}\rangle, \quad a_2 = \langle\lambda\rangle, \quad b_2 = \langle\varepsilon_{nn}\rangle \tag{3.5}$$

Neglecting the influence of the boundary layer [3], we will seek the solution of the system of equations (3.3) – (3.5) with the aid of the Fourier transform [5]. We denote the parameters of the transformation of the variables x_i by ξ_i. The solution of the system of equations (3.3) under the conditions (3.5) has the form

$$\begin{aligned}
\langle\mu\rangle\,\varepsilon_{ij}' = \int \frac{1}{m^2}\Big[\Big(4c\,\langle\varepsilon_{mn}\rangle\frac{\xi_m\xi_n\xi_i\xi_j}{m^2} - 2\,\langle\varepsilon_{ik}\rangle\,\xi_k\xi_j - \\
2\,\langle\varepsilon_{jk}\rangle\,\xi_k\xi_i\Big) f_1 + 2\,\langle\varepsilon_{nn}\rangle\,(c-1)\,\xi_i\xi_j f_2\Big] e^{i\xi_n x_n}\, d\xi
\end{aligned} \tag{3.6}$$
$$c = (\langle\mu\rangle + \langle\lambda\rangle)(2\,\langle\mu\rangle + \langle\lambda\rangle)^{-1}, \qquad m^2 = \xi_i\xi_i$$

Here f_1, f_2 are functions of the variables ξ_i, which determine the spectral decomposition of the random functions μ', λ'

$$\mu' = \int f_1 e^{i\xi_n x_n}\, d\xi, \qquad \lambda' = \int f_2 e^{i\xi_n x_n} d\xi \tag{3.7}$$

The integration is carried out over the entire space of the variables ξ_i. Taking into account the isotropy of the random functions, from the relations (3.6) and (3.7) we obtain [5]

$$\langle\mu\rangle\langle\mu'\varepsilon_{ij}'\rangle = \int\frac{1}{m^2}\left[\left(4c\,\langle\varepsilon_{mn}\rangle\frac{\xi_m\xi_n\xi_i\xi_j}{m^2} - 2\,\langle\varepsilon_{ik}\rangle\,\xi_k\xi_j - 2\,\langle\varepsilon_{jk}\rangle\,\xi_k\xi_i\right)\Phi_1(m) + 2\,\langle\varepsilon_{nn}\rangle(c-1)\,\xi_i\xi_j\Phi_2(m)\right]d\xi \tag{3.8}$$

$$\langle\mu\rangle\langle\lambda'\varepsilon_{nn}'\rangle = 2\,\langle\varepsilon_{nn}\rangle\,(c-1)\,(\langle\lambda'\lambda'\rangle + {}^2/_3\,\langle\lambda'\mu'\rangle)$$

Here $\Phi_1(m)$ is the spectral density of thr random function μ', $\Phi_2(m)$ is the relative spectral density of the random functions λ' and μ' and $\langle\lambda'\lambda'\rangle$ is the dispersion of the random function λ'. Since $\Phi(m)$ is an isotropic function, the integrals which occur in the first relation of (3.8) are isotropic [6, 7], symmetric with respect to all the tensor indices

$$\int\frac{\xi_m\xi_n\xi_i\xi_j}{m^4}\Phi_1(m)\,d\xi = \frac{\langle\mu'\mu'\rangle}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) \tag{3.9}$$

$$\int\frac{\xi_i\xi_j}{m^2}\Phi_1(m)\,d\xi = \frac{\langle\mu'\mu'\rangle}{3}\delta_{ij}, \qquad \int\frac{\xi_i\xi_j}{m^2}\Phi_2(m)\,d\xi = \frac{\langle\mu'\lambda'\rangle}{3}\delta_{ij}$$

Here $\langle\mu'\lambda'\rangle$ is the correlative moment of the random functions μ' and λ' and $\langle\mu'\mu'\rangle$ is the dispersion of the random function μ'. Substituting the expressions (3.8), (3.9) into the first relation of (3.2) we obtain

$$\langle\sigma_{ij}\rangle = 2\mu_1\,\langle\varepsilon_{ij}\rangle + \delta_{ij}\lambda_1\,\langle\varepsilon_{nn}\rangle \tag{3.10}$$

$$\mu_1 = \langle\mu\rangle - 2\,\frac{8\,\langle\mu\rangle + 3\,\langle\lambda\rangle}{15\,\langle\mu\rangle\,(2\,\langle\mu\rangle + \langle\lambda\rangle)}\,\langle\mu'\mu'\rangle \tag{3.11}$$

$$\lambda_1 = \langle\lambda\rangle - \frac{\langle\lambda'\lambda'\rangle - {}^4/_{15}\,(\langle\mu\rangle + \langle\lambda\rangle)\,\langle\mu'\mu'\rangle\,\langle\mu\rangle^{-1} + {}^8/_3\,\langle\mu'\lambda'\rangle}{2\,\langle\mu\rangle + \langle\lambda\rangle}$$

Under the assumption of the noncorrelatedness of the random functions λ' and μ' the expressions (3.11) agree with the results given in [8], where a small fluctuation of the random functions is assumed.

Thus, from the expressions (3.10) it follows that the behavior of the stochastically nonhomogeneous isotropic elastic material is described on the average by the Hooke's law with effective elasticity moduli defined by the relations

$$\gamma = \mu_1, \qquad k = \lambda_1 + {}^2/_3\;\mu_1$$

where γ and k are the shear and volume moduli, respectively.

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COMPARISON OF STATIONARY AND QUASI-STATIONARY STREAMS OF PERFECT FLUID

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The nonstationary stream of perfect fluid whose streamlines are stationary (a quasi-stationary stream) is considered in a conservative field of external forces. Conditions under which a field of unit vectors can simultaneously represent the field of velocity directions of a barotropic quasi-stationary potential or vortex motion of a perfect fluid are determined. Comparison is made with the results cited in [1] for a stationary motion, and the absolute values of velocities of stationary and quasi-stationary streams with common streamlines are compared in certain classes of motion. Comparison is also made with the results presented in [2] for a quasi-stationary stream of a perfect incompressible fluid. The range of the considered classes of unit vector fields, the arbitrariness of determination of the absolute value of velocity, and the acceleration and density potentials for a given velocity direction field are determined. It is assumed that the arbitrariness of solutions is determined in a class of analytic functions and that vector fields are analytic.

1. We denote the unit vector of velocity direction by $\mathbf{e}$ and the vector of streamline curvature by $\mathbf{k}$. A vector field is called holonomic if there exists a set of surfaces orthogonal to it [1]. The quantity $H = \operatorname{div} \mathbf{e}$ is called the mean curvature of field $\mathbf{e}$ [3]. It is assumed that some of the vector lines of the field are not straight.

Any holonomic field $\mathbf{e}$ may be considered to be the velocity direction field of a stationary stream of perfect fluid [4]. For a potential quasi-stationary stream of perfect fluid the statement formulated in Theorem 2 in [2] for such fluid is valid. The geometry of velocity directions of such fields for a perfect incompressible fluid is different, since the incompressibility imposes an additional condition on the velocity direction field (condition 2 in Theorem 1 in [2]). For the quasi-stationary stream of perfect fluid we have the following theorems.

Theorem 1. The absolute values of velocities W and V of a stationary and quasi-stationary streams with common streamlines are related by the expression $V = \psi W$, where ψ is a function which at every instant satisfies the condition $\mathbf{e} \times \operatorname{grad} \psi = 0$.

Since for a perfect incompressible fluid ψ depends only on time (note to Theorem 2 in [2]), hence the arbitrariness of determination of the absolute value of velocity for a